Last time: vector functions

$\vec{r}(t) = \langle f(t), g(t) \rangle$ (2D) $\quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ (3D)

Input: number t $\rightarrow$ Output: 2 numbers / 3 numbers.

Today: Functions of several variables

Input: 2 numbers / 3 numbers $\rightarrow$ Output: a number.

Usually written like...

$f(x,y)$: Takes 2 numbers x, y as inputs
Gives a value $f(x,y)$ as an output

$f(x,y,z)$: Takes 3 numbers x, y, z as inputs
Gives a value $f(x,y,z)$ as an output

Example: $f(x,y) = x+y$, $\quad f(x,y) = x^2 + \sin y$, $\quad f(x,y) = e^{xy}$, ...

$f(1,2) = 1+2 = 3 \quad f(0,\pi) = 0^2 + \sin \pi = 0 \quad f(1,3) = e^3$

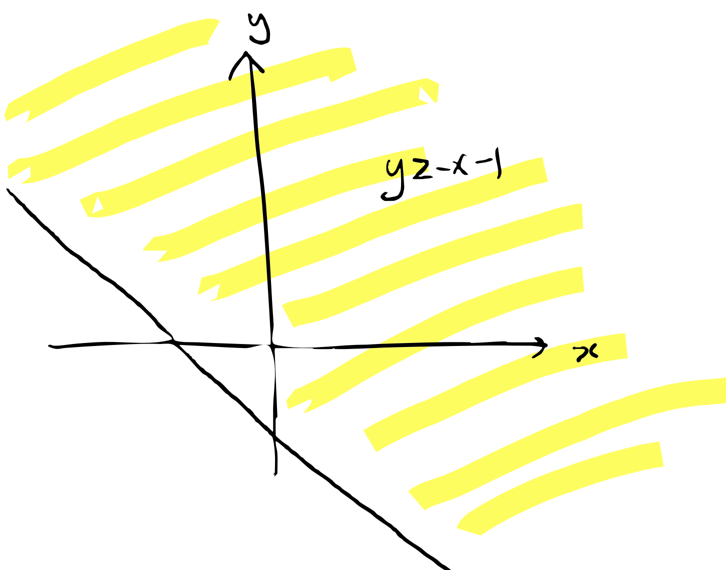
The expression for $f(x,y)$ may make sense at certain parts of the xy-plane.

The region over which the expression makes sense is called the <u>domain</u>. The domains in this case are much more complicated than functions of one variable. This is because the 2D/3D regions are more complicated. When drawing the domain, we draw solid boundary lines if they are included, dotted if not included.

<u>Example</u> $f(x,y) = \sqrt{x+y+1}$

The expression makes sense if $x+y+1 \geq 0$, or $x+y \geq -1$.

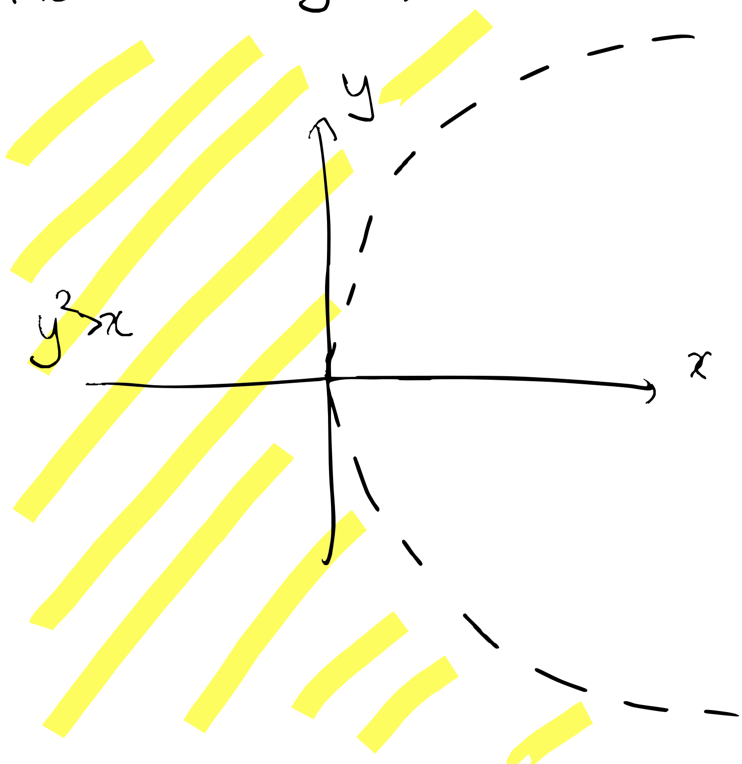
The domain is $\{x+y \geq -1\}$.



Boundary line is $y=-x-1$ and is included in the domain.

<u>Example</u> $f(x,y) = x\ln(y^2-x)$ is defined if $y^2-x>0$.

The domain is $\{y^2 > x\}$.



The dotted boundary line is $y^2 = x$ and is not included.

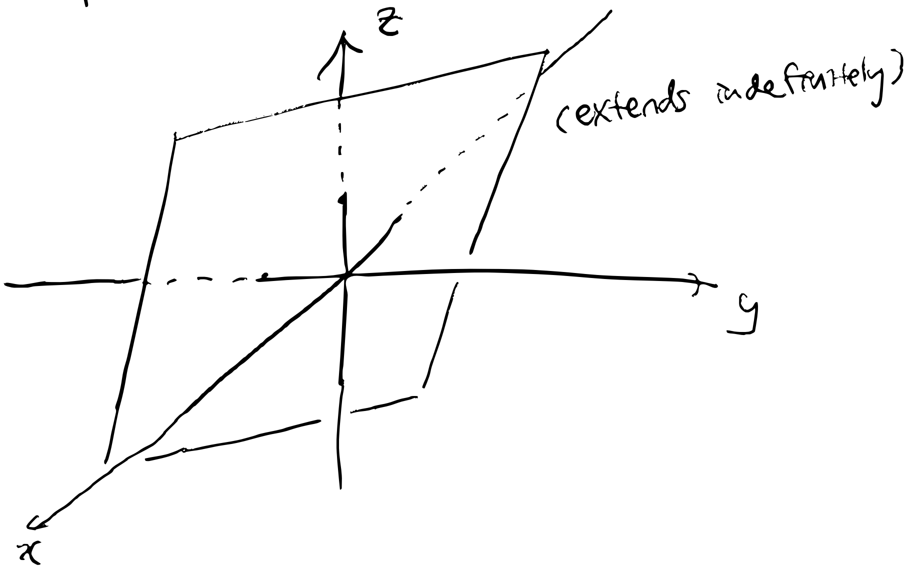
For a function of two variables $f(x,y)$, you can sketch the graph of $z = f(x,y)$.

This is the collection of points

$$\{(x, y, f(x,y))\}.$$

This is a surface in 3D.

<u>Example</u> The graph of $z=f(x,y)$, where $f(x,y)=x+y+1$, is the plane $z=x+y+1$.



<u>Example</u>

The graph of $z=f(x,y)$, where $f(x,y)=\sqrt{9-x^2-y^2}$,

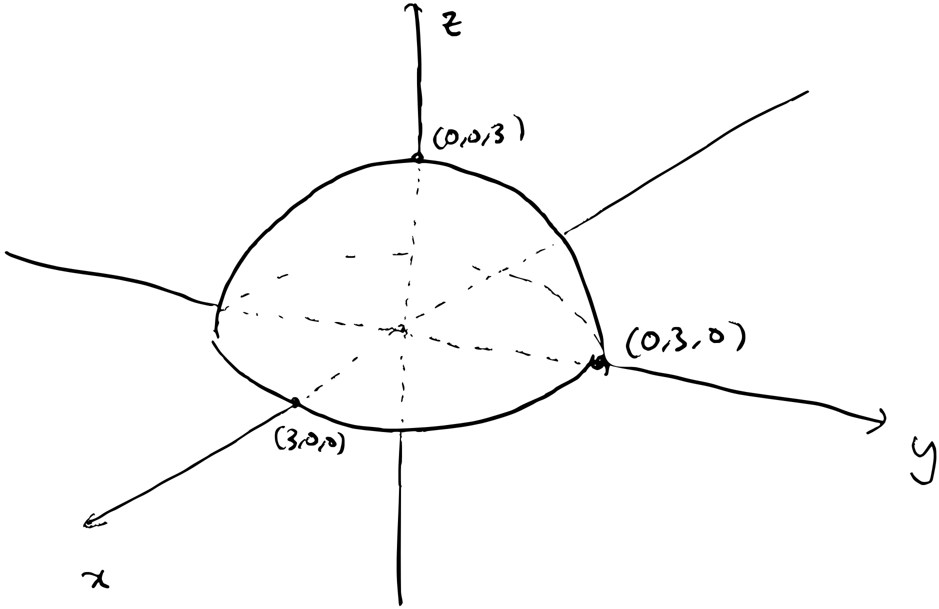
satisfies $z=\sqrt{9-x^2-y^2}$, or $z^2=9-x^2-y^2$, or

$\boxed{x^2+y^2+z^2=9.}$

This is the equation of a sphere but note that the graph does not depict the whole sphere, because

$z=\sqrt{9-x^2-y^2}$ is always $z\geq 0$.

The graph is thus the upper half of the sphere, including the boundary (= equator).



Just like the domains of vector functions, there are some properties that you can check for the domains of functions of several variables.

- **Closed**: when the domain is defined using only $\geq$, $\leq$, $=$ in the expression (NO $>$, $<$, $\neq$)
- **Bounded**: when the domain does not go to infinity and "stay bounded"
- **Compact**: when the domain is both closed and bounded.

Implicit functions

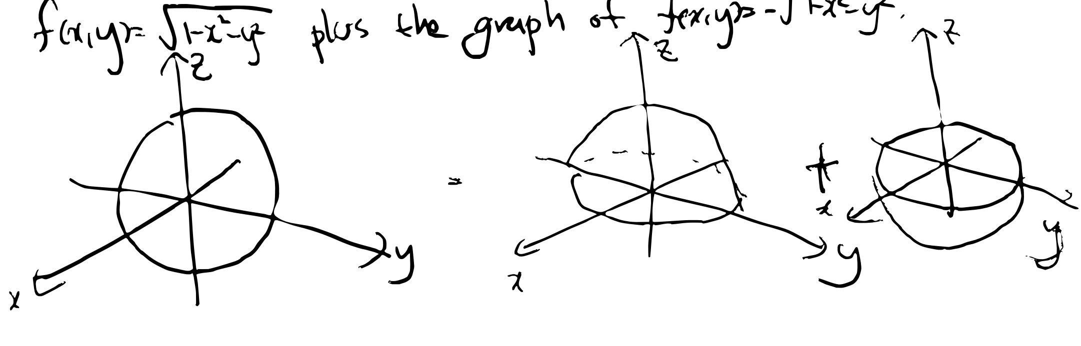
A function can be defined implicitly using an implicit equation involving x, y, z.

<u>Ex</u> The implicit equation $x^2+y^2+z^2=1$ can be regarded as defining an implicit function z in terms of x, y. Namely,

$$z(x,y) = \{\text{The values of } z \text{ such that } x^2+y^2+z^2=1\}.$$

Implicit functions have a peculiar feature that there might be multiple different possible values for z given x, y. In the example of $x^2+y^2+z^2=1$, there are 2 possible values, $\pm\sqrt{1-x^2-y^2}$, and implicit function means you are considering both functions together.

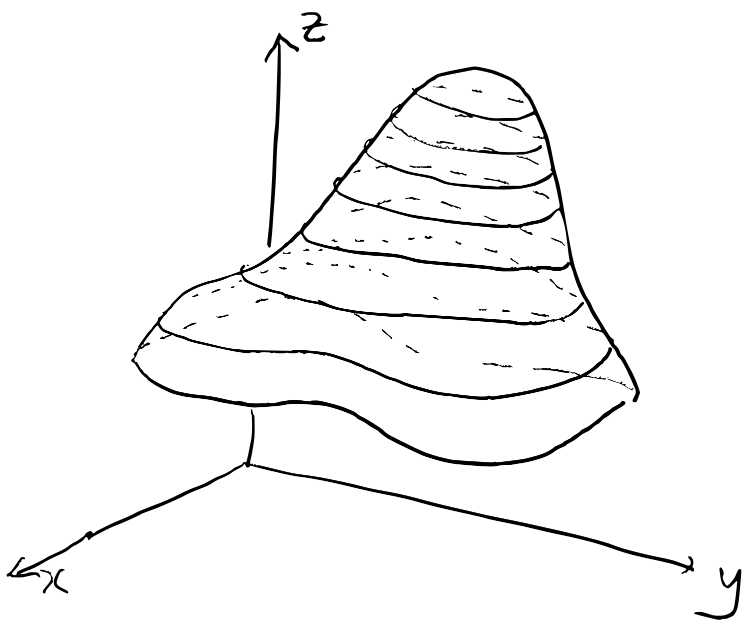
The graph of $x^2+y^2+z^2=1$ therefore is the graph of $f(x,y)=\sqrt{1-x^2-y^2}$ plus the graph of $f(x,y)=-\sqrt{1-x^2-y^2}$.



Horizontal traces and level curves

Since it's difficult to draw a 3D graph on paper, we often put some indicators on the graph to better depict the 3D object.

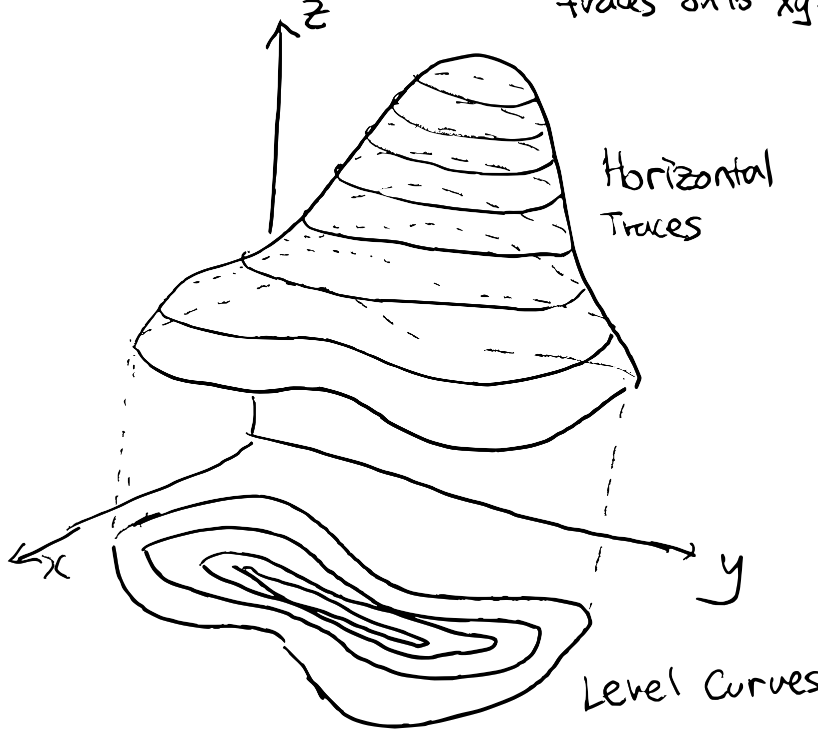
<u>Definition</u> The <u>horizontal trace of level k</u> is the curve in 3D that is the "$z=k$ slice" of the graph. Namely, given an equation for the graph, you set $z=k$. The curve has a fixed z-coordinate, k.



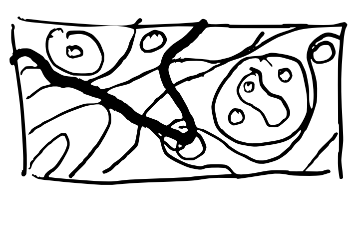
You can even try to encode the data of $f(x,y)$ purely on the xy-plane using this idea.

Definition The <u>level curve of level k</u> is the curve on 2D defined by setting $z=k$ from the equation of the graph. For example, if the graph is $z=f(x,y)$, The level curve of level k is $\{(x,y) \mid f(x,y)=k\}$.

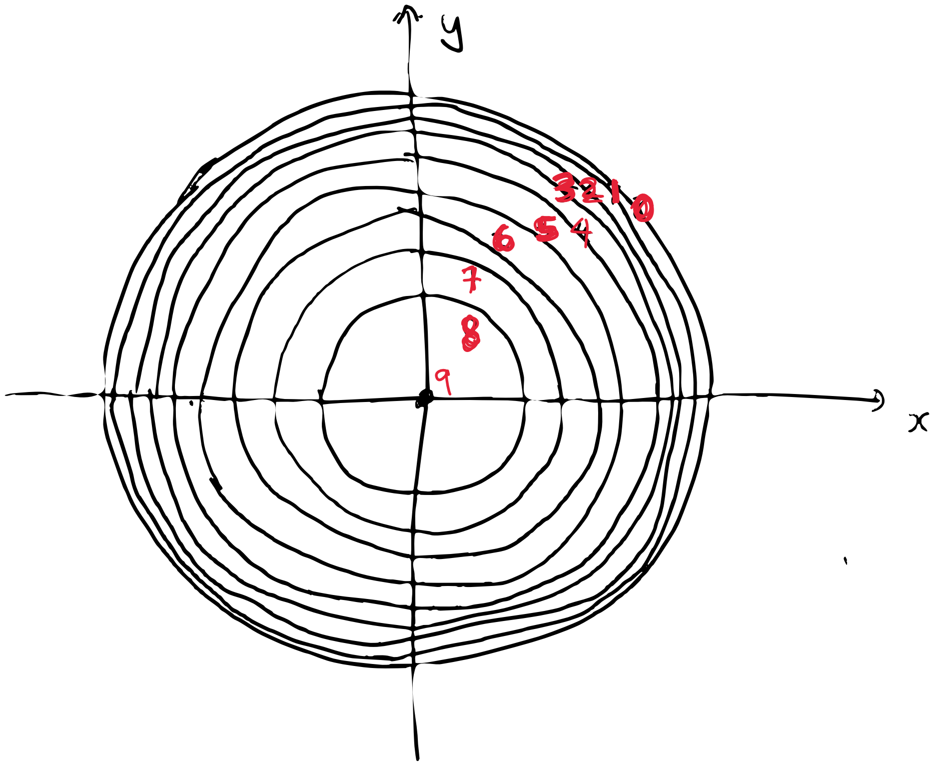
The level curves are the projections of the horizontal traces onto xy-plane!



A collection of level curves on 2D is called a <u>contour map</u>. Example) Topographic maps, weather forecast maps.



Example From the graph of $z = \sqrt{9 - x^2 - y^2}$, the contour map looks like:



The numbers on the level curves indicate the levels.

Or, you may use different colors for different level curves.